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1990 J. Phys. A: Math. Gen. 23 4127

(http://iopscience.iop.org/0305-4470/23/18/018)

J. Phys. A: Math. Gen. 23 (1990) 4127-4135. Printed in the UK

Aharonov–Bohm scattering on two parallel flux tubes of the same magnetic flux

Zhi-Yu Gu†‡ and Shang-Wu Qian†§||

† Center of Theoretical Physics, CCAST (World Laboratory), Beijing 100080, People's Republic of China

‡ Beijing Teacher's College (Division), Bai Guang Street, Beijing 100053, People's Republic of China

§ Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China

|| Department of Physics, Peking University, Beijing 100871, People's Republic of China

Received 30 August 1989, in final form 25 May 1990

Abstract. The problem of Aharonov–Bohm scattering on two parallel flux tubes of the same magnetic flux is solved exactly and the differential cross section is calculated.

1. Introduction

In our previous paper [2], we have solved exactly the Aharonov–Bohm (AB) scattering on two parallel flux lines of the same magnitude. In order to study the core size effect, in this paper we shall further solve exactly the AB scattering on two parallel flux tubes of the same magnitude Φ. The AB scattering on a single flux tube has previously been analysed by Aharonov *et al* [1]. We shall use the results of [2] and [1] to discuss our problem.

2. The wavefunction

Let $0XY$ be the coordinate plane perpendicular to two flux tubes of radius R. The coordinates of the tube centres are $(a, 0)$ and $(-a, 0)$. In the case that the magnetic field is uniformly distributed in the tubes, the vector potential in the outside region is the same as in the case of the flux lines. Hence, we can directly write down the expression for the wavefunction, which is just the corrected version of equation (23) of [2] (as discussed in the corrigendum of [2]):

$$\begin{aligned}\psi = \sum_{m=0}^{\infty} \sum_{l} \{&[C^c_{ml} + c^c_{ml} q + \mathrm{O}(q^2)] Ce_l(\mu, q) + [\bar{C}^c_{ml} + \bar{c}^c_{ml} q + \mathrm{O}(q^2)] Fey_l(\mu, q) \\ &+ [S^c_{ml} + s^c_{ml} q + \mathrm{O}(q^2)] Se_l(\mu, q) \\ &+ [\bar{S}^c_{ml} + \bar{s}^c_{ml} q + \mathrm{O}(q^2)] Gey_l(\mu, q)\} ce_m(\theta, q) \\ &+ \sum_{m=1}^{\infty} \sum_{l} \{[C^s_{ml} + c^s_{ml} q + \mathrm{O}(q^2)] Ce_l(\mu, q) \\ &+ [\bar{C}^s_{ml} + \bar{c}^s_{ml} q + \mathrm{O}(q^2)] Fey_l(\mu, q) \\ &+ [S^s_{ml} + s^s_{ml} q + \mathrm{O}(q^2)] Se_l(\mu, q) \\ &+ [\bar{S}^s_{ml} + \bar{s}^s_{ml} q + \mathrm{O}(q^2)] Gey_l(\mu, q)\} se_m(\theta, q)\end{aligned} \tag{1}$$

where (μ, θ) are elliptical coordinates, $q \equiv a^2k^2/4$, $k \equiv (2mE/\hbar^2)^{1/2}$ is the wave number and $\alpha = -e\Phi/2\pi\hbar c$ is the quantum number of the flux. The coefficients C^c_{ml}, $\bar{C}^c_{ml}$, $S^c_{ml}, \ldots$ are functions of α. When $a \to 0$, that is when $q \to 0$, two magnetic tubes of flux Φ become one magnetic flux tube of flux 2Φ. The solutions of these two cases should be equal and we shall use this fact to determine the coefficients C^c_{ml}, $\bar{C}^c_{ml}$, $S^c_{ml}, \ldots$. For the case of one impenetrable tube with flux 2Φ, we can use the results of [1], and directly write out the expression for the wavefunction:

$$\psi = \sum_{m=-\infty}^{\infty} \phi_{mk}(\rho)\, \mathrm{e}^{im\phi} \tag{2}$$

where

$$\phi_{mk}(\rho) = A_m(k, R, 2\alpha)[Y_{|m+2\alpha|}(kR)J_{|m+2\alpha|}(k\rho) - J_{|m+2\alpha|}(kR)Y_{|m+2\alpha|}(k\rho)] \tag{3}$$

and

$$A_m(k, R, 2\alpha) = \frac{(-\mathrm{i})^{|2\alpha - m(2\tau/\pi)|-1}}{H^{(1)}_{|m+2\alpha|}(kR)} \tag{4}$$

in [1], $\tau = -\pi/2$. Apparently, $\phi_{mk}(R) = 0$, that is the required boundary condition is satisfied. Expression (2) can be written as

$$\begin{aligned}\psi = \sum_{m=0}^{\infty} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi + \mathrm{i}m\tau}}{H^{(1)}_{m+2\alpha}(kR)}[Y_{m+2\alpha}(kR)J_{m+2\alpha}(k\rho) - J_{m+2\alpha}(kR)Y_{m+2\alpha}(k\rho)]\,\mathrm{e}^{\mathrm{i}m\phi} \\ + \sum_{m=1}^{\infty} \frac{\mathrm{i}(-1)^m\,\mathrm{e}^{\mathrm{i}\alpha\pi - \mathrm{i}m\tau}}{H^{(1)}_{m-2\alpha}(kR)}[Y_{m-2\alpha}(kR)J_{m-2\alpha}(k\rho) \\ - J_{m-2\alpha}(kR)Y_{m-2\alpha}(k\rho)]\,\mathrm{e}^{-\mathrm{i}m\phi}.\end{aligned} \tag{5}$$

In the asymptotic region ($\rho \to \infty$ or $\mu \to \infty$), the Bessel functions

$$\begin{aligned}J_{m\pm 2\alpha}(k\rho) &= \cos\alpha\pi\, J_m(k\rho) \pm \sin\alpha\pi\, Y_m(k\rho) \\ Y_{m\pm 2\alpha}(k\rho) &= \mp\sin\alpha\pi\, J_m(k\rho) + \cos\alpha\pi\, Y_m(k\rho).\end{aligned} \tag{6}$$

Substituting (6) into (5) we obtain

$$\begin{aligned}\psi = \sum_{m=0}^{\infty} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi + \mathrm{i}m\tau}}{H^{(1)}_{m+2\alpha}(kR)}\{Y_{m+2\alpha}(kR)[\cos\alpha\pi\, J_m(k\rho) + \sin\alpha\pi\, Y_m(k\rho)] \\ - J_{m+2\alpha}(kR)[-\sin\alpha\pi\, J_m(k\rho) + \cos\alpha\pi\, Y_m(k\rho)]\}(\cos m\phi + \mathrm{i}\sin m\phi) \\ + \sum_{m=1}^{\infty} \frac{\mathrm{i}(-1)^m\,\mathrm{e}^{\mathrm{i}\alpha\pi - \mathrm{i}m\tau}}{H^{(1)}_{m-2\alpha}(kR)}\{Y_{m-2\alpha}(kR)[\cos\alpha\pi\, J_m(k\rho) - \sin\alpha\pi\, Y_m(k\rho)] \\ - J_{m-2\alpha}(kR)[\sin\alpha\pi\, J_m(k\rho) + \cos\alpha\pi\, Y_m(k\rho)]\}(\cos m\phi - \mathrm{i}\sin m\phi) \\ = H^J_{c0}J_0(k\rho) + H^Y_{c0}Y_0(k\rho) + \sum_{n=1}^{\infty}[H^J_{c2n}J_{2n}(k\rho) + H^Y_{c2n}Y_{2n}(k\rho)]\cos 2n\phi \\ + \sum_{n=0}^{\infty}[H^J_{c2n+1}J_{2n+1}(k\rho) + H^Y_{c2n+1}Y_{2n+1}(k\rho)]\cos(2n+1)\phi \\ + \sum_{n=0}^{\infty}[H^J_{s2n+1}J_{2n+1}(k\rho) + H^Y_{s2n+1}Y_{2n+1}(k\rho)]\sin(2n+1)\phi \\ + \sum_{n=0}^{\infty}[H^J_{s2n+2}J_{2n+2}(k\rho) + H^Y_{s2n+2}Y_{2n+2}(k\rho)]\sin(2n+2)\phi\end{aligned} \tag{7}$$

where

$$H^{J}_{c0}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi}}{H^{(1)}_{2\alpha}(kR)}[Y_{2\alpha}(kR)\cos\alpha\pi+J_{2\alpha}(kR)\sin\alpha\pi]$$

$$H^{Y}_{c0}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi}}{H^{(1)}_{2\alpha}(kR)}[Y_{2\alpha}(kR)\sin\alpha\pi-J_{2\alpha}(kR)\cos\alpha\pi]$$

$$H^{J}_{c2n}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}2n\tau}}{H^{(1)}_{2n+2\alpha}(kR)}[Y_{2n+2\alpha}(kR)\cos\alpha\pi+J_{2n+2\alpha}(kR)\sin\alpha\pi]$$
$$+\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}2n\tau}}{H^{(1)}_{2n-2\alpha}(kR)}[Y_{2n-2\alpha}(kR)\cos\alpha\pi-J_{2n-2\alpha}(kR)\sin\alpha\pi]$$

$$H^{Y}_{c2n}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}2n\tau}}{H^{(1)}_{2n+2\alpha}(kR)}[Y_{2n+2\alpha}(kR)\sin\alpha\pi-J_{2n+2\alpha}(kR)\cos\alpha\pi]$$
$$+\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}2n\tau}}{H^{(1)}_{2n-2\alpha}(kR)}[-Y_{2n-2\alpha}(kR)\sin\alpha\pi-J_{2n-2\alpha}(kR)\cos\alpha\pi]$$

$$H^{J}_{c2n+1}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1+2\alpha}(kR)}[Y_{2n+1+2\alpha}(kR)\cos\alpha\pi+J_{2n+1+2\alpha}(kR)\sin\alpha\pi]$$
$$-\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1-2\alpha}(kR)}[Y_{2n+1-2\alpha}(kR)\cos\alpha\pi-J_{2n+1-2\alpha}(kR)\sin\alpha\pi]$$

$$H^{Y}_{c2n+1}=\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1+2\alpha}(kR)}[Y_{2n+1+2\alpha}(kR)\sin\alpha\pi-J_{2n+1+2\alpha}(kR)\cos\alpha\pi]$$
$$-\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1-2\alpha}(kR)}[-Y_{2n+1-2\alpha}(kR)\sin\alpha\pi-J_{2n+1-2\alpha}(kR)\cos\alpha\pi]$$

$$H^{J}_{s2n+1}=\frac{-\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1+2\alpha}(kR)}[Y_{2n+1+2\alpha}(kR)\cos\alpha\pi-J_{2n+1+2\alpha}(kR)\sin\alpha\pi]$$
$$-\frac{\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1-2\alpha}(kR)}[Y_{2n+1-2\alpha}(kR)\cos\alpha\pi-J_{2n+1-2\alpha}(kR)\sin\alpha\pi]$$

$$H^{Y}_{s2n+1}=\frac{-\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1+2\alpha}(kR)}[Y_{2n+1+2\alpha}(kR)\sin\alpha\pi-J_{2n+1+2\alpha}(kR)\cos\alpha\pi]$$
$$-\frac{\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+1)\tau}}{H^{(1)}_{2n+1-2\alpha}(kR)}[-Y_{2n+1-2\alpha}(kR)\sin\alpha\pi-J_{2n+1-2\alpha}(kR)\cos\alpha\pi]$$

$$H^{J}_{s2n+2}=\frac{-\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+2)\tau}}{H^{(1)}_{2n+2+2\alpha}(kR)}[Y_{2n+2+2\alpha}(kR)\cos\alpha\pi+J_{2n+2+2\alpha}(kR)\sin\alpha\pi]$$
$$+\frac{\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+2)\tau}}{H^{(1)}_{2n+2-2\alpha}(kR)}[Y_{2n+2-2\alpha}(kR)\cos\alpha\pi-J_{2n+2-2\alpha}(kR)\sin\alpha\pi]$$

$$H^{Y}_{s2n+2}=\frac{-\mathrm{e}^{-\mathrm{i}\alpha\pi+\mathrm{i}(2n+2)\tau}}{H^{(1)}_{2n+2+2\alpha}(kR)}[Y_{2n+2+2\alpha}(kR)\sin\alpha\pi-J_{2n+2+2\alpha}(kR)\cos\alpha\pi]$$
$$+\frac{\mathrm{e}^{\mathrm{i}\alpha\pi-\mathrm{i}(2n+2)\tau}}{H^{(1)}_{2n+2-2\alpha}(kR)}[-Y_{2n+2-2\alpha}(kR)\sin\alpha\pi-J_{2n+2-2\alpha}(kR)\cos\alpha\pi]. \tag{8}$$

When $\mu\to\infty$, $q\to 0$, (1) becomes

$$\psi=\sum_{m=0}^{\infty}\sum_{l}[C^c_{ml}p'_lJ_l(k\rho)+\bar{C}^c_{ml}p'_lY_l(k\rho)$$
$$+S^c_{ml}s'_lJ_l(k\rho)+\bar{S}^c_{ml}s'_lY_l(k\rho)]\cos m\phi$$
$$+\sum_{m=1}^{\infty}\sum_{l}[C^s_{ml}p'_lJ_l(k\rho)+\bar{C}^s_{ml}p'_lY_l(k\rho)$$
$$+S^s_{ml}s'_lJ_l(k\rho)+\bar{S}^s_{ml}s'_lY_l(k\rho)]\sin m\phi \tag{9}$$

where the constant multipliers p'_l and s'_l are given in Mclachlan's book [3, pp 368-9]. Comparing (9) with (7) we obtain the coefficients C^c_{ml}, $\bar{C}^c_{ml}$, $S^c_{ml}, \ldots$; finally we obtain the expression for the wavefunction in the case of two impenetrable flux tubes:

$$\psi=\left([H^J_{c0}+h^J_{c0}q+O(q^2)]\frac{Ce_0(\mu,q)}{p'_0}\right.$$
$$\left.+[H^Y_{c0}+h^Y_{c0}q+O(q^2)]\frac{Fey_0(\mu,q)}{p'_0}\right)ce_0(\theta,q)$$
$$+\sum_{n=1}^{\infty}\left([H^J_{c2n}+h^J_{c2n}q+O(q^2)]\frac{Ce_{2n}(\mu,q)}{p'_{2n}}\right.$$
$$\left.+[H^Y_{c2n}+h^Y_{c2n}q+O(q^2)]\frac{Fey_{2n}(\mu,q)}{p'_{2n}}\right)ce_{2n}(\theta,q)$$
$$+\sum_{n=0}^{\infty}\left([H^J_{c2n+1}+h^J_{c2n+1}q+O(q^2)]\frac{Ce_{2n+1}(\mu,q)}{p'_{2n+1}}\right.$$
$$\left.+[H^Y_{c2n+1}+h^Y_{c2n+1}q+O(q^2)]\frac{Fey_{2n+1}(\mu,q)}{p'_{2n+1}}\right)ce_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}\left([H^J_{s2n+1}+h^J_{s2n+1}q+O(q^2)]\frac{Se_{2n+1}(\mu,q)}{s'_{2n+1}}\right.$$
$$\left.+[H^Y_{s2n+1}+h^Y_{s2n+1}q+O(q^2)]\frac{Gey_{2n+1}(\mu,q)}{s'_{2n+1}}\right)se_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}\left([H^J_{s2n+2}+h^J_{s2n+2}q+O(q^2)]\frac{Se_{2n+2}(\mu,q)}{s'_{2n+2}}\right.$$
$$\left.+[H^Y_{s2n+2}+h^Y_{s2n+2}q+O(q^2)]\frac{Gey_{2n+2}(\mu,q)}{s'_{2n+2}}\right)se_{2n+2}(\theta,q). \tag{10}$$

3. The scattering cross section

In the asymptotic region $\phi=\theta$,

$$\psi=\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]+f(R,\theta)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}. \tag{11}$$

By the orthogonality of Mathieu functions we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]y_i(\theta,q)\,\mathrm{d}\theta+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)y_j(\theta,q)\,\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\psi y_j(\theta,q)\,\mathrm{d}\theta \qquad j=0,1,2,3,4 \tag{12}$$

where $y_0(\theta,q)=ce_0(\theta,q)$, $y_1(\theta,q)=ce_{2n}(\theta,q)$, $y_2(\theta,q)=ce_{2n+1}(\theta,q)$ $y_3(\theta,q)=se_{2n+1}(\theta,q)$ and $y_4(\theta,q)=se_{2n+2}(\theta,q)$. The terms in (12) can be rewritten as

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]y_j(\theta,q)\,\mathrm{d}\theta=G_j$$
$$=G_j^+(\alpha,q)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}+G_j^-(\alpha,q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{13}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)y_j(\theta,q)\,\mathrm{d}\theta=F_j \tag{14}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\psi y_j(\theta,q)\,\mathrm{d}\theta=H_{Rj}$$
$$=[H_{Rj}^++h_{Rj}^+q+\mathrm{O}(q^2)]\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}+[H_{Rj}^-+h_{Rj}^-q+\mathrm{O}(q^2)]\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}. \tag{15}$$

Substituting (13)-(15) into (12), then comparing the coefficients of $\mathrm{e}^{\mathrm{i}k\rho}/\sqrt{k\rho}$, we can find $F_j(\alpha,\tau)$:

$$F_j=H_{Rj}^+-G_j^+. \tag{16}$$

Since $y_j(\theta,q)$ form a complete set, we can express $f(\theta)$ as

$$f(\theta)=\tfrac{1}{2}F_0ce_0(\theta,q)+\sum_{n=1}^{\infty}F_1ce_{2n}(\theta,q)+\sum_{n=0}^{\infty}F_2ce_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}F_3se_{2n+1}(\theta,q)+\sum_{n=0}^{\infty}F_4se_{2n+2}(\theta,q) \tag{17}$$

the coefficient of $\mathrm{e}^{-\mathrm{i}k\rho}/\sqrt{k\rho}$ can be proved to be equal to zero. The sum of those terms containing G_j^+ in (17) equals zero, hence

$$f(\theta,R,q)=\tfrac{1}{2}[H_{R0}^++h_{R0}^+q+\mathrm{O}(q^2)]ce_0(\theta,q)$$
$$+\sum_{n=1}^{\infty}[H_{R1}^++h_{R1}^+q+\mathrm{O}(q^2)]ce_{2n}(\theta,q)$$
$$+\sum_{n=0}^{\infty}[H_{R2}^++h_{R2}^+q+\mathrm{O}(q^2)]ce_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}[H_{R3}^++h_{R3}^+q+\mathrm{O}(q^2)]se_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}[H_{R4}^++h_{R4}^+q+\mathrm{O}(q^2)]se_{2n+2}(\theta,q). \tag{18}$$

3.1. Calculation of H_{Rj}

In the asymptotic region

$$\begin{aligned} J_m(k\rho) &\sim [\mathrm{e}^{\mathrm{i}(k\rho - m\pi/2-\pi/4)} + \mathrm{e}^{-\mathrm{i}(k\rho - m\pi/2-\pi/4)}]/(2\pi k\rho) \\ Y_m(k\rho) &\sim [\mathrm{e}^{\mathrm{i}(k\rho - m\pi/2-\pi/4)} - \mathrm{e}^{-\mathrm{i}(k\rho - m\pi/2-\pi/4)}]/\mathrm{i}(2\pi k\rho) \\ H_m^{(1)}(k\rho) &\sim 2\,\mathrm{e}^{\mathrm{i}(k\rho - m\pi/2-\pi/4)}/(2\pi k\rho). \end{aligned} \tag{19}$$

Substituting (19) into (10) and (15) we obtain

$$H_{R0} = \left([H_0^+ + h_0^+ q + \mathrm{O}(q^2)] - \frac{2\,\mathrm{e}^{-\mathrm{i}2\alpha\pi - \mathrm{i}\pi/4}}{\sqrt{2\pi}}\, 2\, \frac{J_{2\alpha}(kR)}{H_{2\alpha}^{(1)}(kR)}\right) \times \frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_0^-(\alpha, q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{20}$$

$$H_{R1} = \left([H_1^+ + h_1^+ q + \mathrm{O}(q^2)] - \frac{2\,\mathrm{e}^{-\mathrm{i}2\alpha\pi + \mathrm{i}2n\tau - \mathrm{i}\pi/4}(-1)^n}{\sqrt{2\pi}} \frac{J_{2n+2\alpha}(kR)}{H_{2n+2\alpha}^{(1)}(kR)} - \frac{2(-1)^n\,\mathrm{e}^{\mathrm{i}2\alpha\pi - \mathrm{i}2n\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n-2\alpha}(kR)}{H_{2n-2\alpha}^{(1)}(kR)}\right)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_1^-(\alpha, q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{21}$$

$$H_{R2} = \left([H_2^+ + h_2^+ q + \mathrm{O}(q^2)] - \frac{2\mathrm{i}(-1)^n\,\mathrm{e}^{-\mathrm{i}2\alpha\pi + \mathrm{i}(2n+1)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+1+2\alpha}(kR)}{H_{2n+1+2\alpha}^{(1)}(kR)} + \frac{2\mathrm{i}(-1)^n\,\mathrm{e}^{\mathrm{i}2\alpha\pi + \mathrm{i}(2n+1)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+1-2\alpha}(kR)}{H_{2n+1-2\alpha}^{(1)}(kR)}\right)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_2^-(\alpha, q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{22}$$

$$H_{R3} = \left([H_3^+ + h_3^+ q + \mathrm{O}(q^2)] - \frac{2(-1)^n\,\mathrm{e}^{-\mathrm{i}2\alpha\pi + \mathrm{i}(2n+1)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+1+2\alpha}(kR)}{H_{2n+1+2\alpha}^{(1)}(kR)} - \frac{2(-1)^n\,\mathrm{e}^{\mathrm{i}2\alpha\pi - \mathrm{i}(2n+1)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+1-2\alpha}(kR)}{H_{2n+1-2\alpha}^{(1)}(kR)}\right)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_3^-(\alpha, q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{23}$$

$$H_{R4} = \left([H_4^+ + h_4^+ q + \mathrm{O}(q^2)] - \frac{2\mathrm{i}(-1)^{n+1}\,\mathrm{e}^{-\mathrm{i}2\alpha\pi + \mathrm{i}(2n+2)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+2+2\alpha}(kR)}{H_{2n+2+2\alpha}^{(1)}(kR)} + \frac{2\mathrm{i}(-1)^n\,\mathrm{e}^{\mathrm{i}2\alpha\pi - \mathrm{i}(2n+2)\tau - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2n+2-2\alpha}(kR)}{H_{2n+2-2\alpha}^{(1)}(kR)}\right)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_4^-(\alpha, q)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}. \tag{24}$$

Substituting $H_{Rj}^+ + h_{Rj}^+ q + \mathrm{O}(q^2)$ (i.e. the coefficient of $\mathrm{e}^{\mathrm{i}k\rho}/\sqrt{k\rho}$ in H_{Rj}) of (20)-(24) into (18), and noting that the sum of all the terms containing H_j^+ is just the scattering cross section $f(\theta, 0, q)$ for the case $R = 0$, we finally obtain

$$\begin{aligned} f(\theta, R, q) &= f(\theta, 0, q) + f_R \\ &= f(\theta, 0, q) + \frac{2\,\mathrm{e}^{-\mathrm{i}2\alpha\pi - \mathrm{i}\pi/4}}{\sqrt{2\pi}} \frac{J_{2\alpha}(kR)}{H_{2\alpha}^{(1)}(kR)}\, ce_0(\theta, q) \\ &\quad + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\,\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}} \left(\mathrm{e}^{-\mathrm{i}2\alpha\pi + \mathrm{i}2n\tau} \frac{J_{2n+2\alpha}(kR)}{H_{2n+2\alpha}^{(1)}(kR)} + \mathrm{e}^{\mathrm{i}2\alpha\pi - \mathrm{i}2n\tau} \frac{J_{2n-2\alpha}(kR)}{H_{2n-2\alpha}^{(1)}(kR)}\right) ce_{2n}(\theta, q) \end{aligned}$$

$$+\sum_{n=0}^{\infty}\frac{2\mathrm{i}(-1)^{n+1}\,\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\left(\mathrm{e}^{-\mathrm{i}2\alpha\pi+\mathrm{i}(2n+1)\tau}\frac{J_{2n+1+2\alpha}(kR)}{H^{(1)}_{2n+1+2\alpha}(kR)}\right.$$

$$\left.-\,\mathrm{e}^{\mathrm{i}2\alpha\pi-\mathrm{i}(2n+1)\tau}\frac{J_{2n+1-2\alpha}(kR)}{H^{(1)}_{2n+1-2\alpha}(kR)}\right)ce_{2n+1}(\theta,q)$$

$$+\sum_{n=0}^{\infty}\frac{2(-1)^{n+1}\,\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\left(\mathrm{e}^{-\mathrm{i}2\alpha\pi+\mathrm{i}(2n+1)\tau}\frac{J_{2n+1+2\alpha}(kR)}{H^{(1)}_{2n+1+2\alpha}(kR)}\right.$$

$$\left.-\,\mathrm{e}^{\mathrm{i}2\alpha\pi-\mathrm{i}(2n+1)\tau}\frac{J_{2n+1-2\alpha}(kR)}{H^{(1)}_{2n+1-2\alpha}(kR)}\right)se_{2n+1}(\theta,q)$$

$$+\sum_{n=0}^{\infty}\frac{2\mathrm{i}(-1)^{n}\,\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\left(\mathrm{e}^{-\mathrm{i}2\alpha\pi+\mathrm{i}(2n+2)\tau}\frac{J_{2n+2+2\alpha}(kR)}{H^{(1)}_{2n+2+2\alpha}(kR)}\right.$$

$$\left.-\,\mathrm{e}^{\mathrm{i}2\alpha\pi-\mathrm{i}(2n+2)\tau}\frac{J_{2n+2-2\alpha}(kR)}{H^{(1)}_{2n+2-2\alpha}(kR)}\right)se_{2n+2}(\theta,q). \tag{25}$$

3.2. *The case when q is small*

In this case we can expand $y_j(\theta, q)$ as a power series of q, and also $f(\theta, 0, q)$ and f_R:

$$f(\theta,0,q)=f_0(\theta)+f_1(\theta)+\mathrm{O}(q^2). \tag{26}$$

From [2] we know that

$$f_0(\theta)=\frac{\mathrm{e}^{-\mathrm{i}3\pi/4}}{\sqrt{2\pi}}\sin 2\alpha\pi\,\exp\left[-\mathrm{i}\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right)\right]\left[\cos\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right)\right]^{-1}. \tag{27}$$

From the corrigendum of [2] we know the corrected version of $f_1(\theta)$:

$$f_1(\theta)=\frac{q\,\mathrm{e}^{-\mathrm{i}\pi/4}}{2\sqrt{2\pi}}\sin(2\alpha\pi)\{\mathrm{i}\cos(2\theta)+\sin(2\theta)+\cos(\tau-\theta)$$

$$\times[\cosh^{-1}|\sec(\tau+\theta)|+\ln|2\cos(\tau+\theta)|]\}. \tag{28}$$

This is just the equation (E6) in the corrigendum of [2].

Similar to (26) we have

$$f_R=f_{R0}+f_{R1}+\mathrm{O}(q^2). \tag{29}$$

Through calculation we obtain the term not containing q,

$$f_{R0}=-\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}\pi/4}\frac{2\,\mathrm{e}^{2\mathrm{i}\delta_m(\alpha)}J_{|m+2\alpha|}(kR)}{H^{(1)}_{|m+2\alpha|}(kR)}\mathrm{e}^{\mathrm{i}m\theta} \tag{30}$$

where

$$\delta_m(\alpha)=\begin{cases}-\alpha\pi+\frac{1}{2}m(\tau+\frac{1}{2}\pi) & \text{when } m\geqslant 0\\ \alpha\pi+\frac{1}{2}m(\tau+\frac{1}{2}\pi) & \text{when } m<0\end{cases} \tag{31}$$

and the term containing the first power of q

$$f_{R1}=q\sum_{m=-\infty}^{\infty}\frac{\mathrm{i}^m\,\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\frac{2\,\mathrm{e}^{2\mathrm{i}\delta_m(\alpha)}J_{|m+2\alpha|}(kR)}{H^{(1)}_{|m+2\alpha|}(kR)}\left(\frac{\mathrm{e}^{\mathrm{i}(m+2)\theta}}{4(m+1)}-\frac{\mathrm{e}^{\mathrm{i}(m-2)\theta}}{4(m-1)}\right). \tag{32}$$

Substituting (26)-(32) into (25) we obtain

$$f(\theta,R,q)=f_0(\theta)+f_1(\theta)+f_{R0}+f_{R1}+\mathrm{O}(q^2). \tag{33}$$

When $q=0$ we get

$$f(\theta, R, 0) = f_0(\theta) + f_{R0} \tag{34}$$

the same result as that in [1].

3.3. *The case when $R \ll a$ and $q \equiv a^2k^2/4$ is yet very small*

On the RHS of (30), the order of magnitude for all terms $m \neq 0$ is (see equation (25) of [1])

$$O((kR)^{2|m|+2\alpha\,\mathrm{sgn}(m)}). \tag{35}$$

They make a very small contribution to the sum, the $m=0$ term making the chief contribution. Calculating the $m=0$ term and comparing it with equation (39) of [1], we obtain

$$\begin{aligned}\lim_{kR\to 0} f_{R0} &= -\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\mathrm{i}\pi/4-\mathrm{i}2\alpha\pi}\,\frac{J_{|2\alpha|}(kR)}{H^{(1)}_{|2\alpha|}(kR)}\\ &= \sqrt{2\pi}\,\mathrm{e}^{\mathrm{i}\pi/4-\mathrm{i}\alpha\pi}\,\frac{\alpha(kR/2)^{2\alpha}}{\Gamma(1+2\alpha)}\end{aligned} \tag{36}$$

and

$$\lim_{kR\to 0} f_{R1} = -q\sqrt{\frac{\pi}{2}}\,\mathrm{e}^{-\mathrm{i}\pi(\alpha-1/4)}\,\frac{\alpha(kR/2)^{2\alpha}}{\Gamma(1+2\alpha)}\cos 2\theta. \tag{37}$$

Substituting (36) and (37) into (33) and neglecting terms containing $O(q^2)$ and

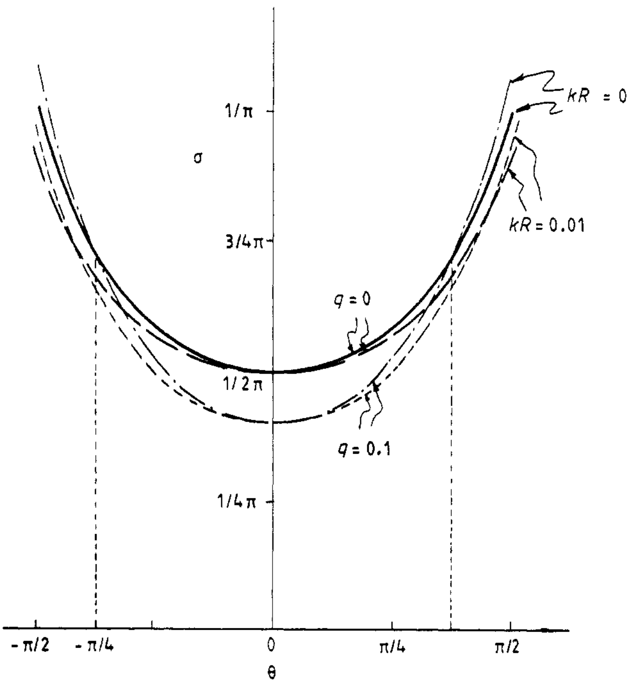

Figure 1. Dependence of σ on θ.

$O((kR)^{2\alpha})$ we obtain

$$f(\theta, R, q) = f_0(\theta) + f_1(\theta) + \sqrt{2\pi}\, e^{i\pi/4 - i\alpha\pi} \frac{\alpha(kR/2)^{2\alpha}}{\Gamma(1+2\alpha)} \left(1 - \frac{q}{2}\cos 2\theta\right) \tag{38}$$

and

$$\sigma = |f(\theta, R, q)|^2$$
$$= \sigma_{R=0} - \frac{2\alpha(kR/2)^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\sin 2\alpha\pi \cos[\frac{1}{2}(\theta+\tau) + \frac{1}{4}\pi - \alpha\pi]}{\cos[\frac{1}{2}(\theta+\tau) + \frac{1}{4}\pi]} \left(1 - \frac{q}{2}\cos 2\theta\right). \tag{39}$$

When $\tau = -\pi/2$, $\alpha = \frac{1}{4}$ the dependence of σ on θ for $q = 0$ and $q = 0.1$, $kR = 0$ and $kR = 0.01$ are shown in figure 1.

Acknowledgments

We would like to thank the referees for valuable suggestions and comments. We are grateful to Professor Murray Peshkin and Professor D H Kobe for many helpful discussions and great encouragement. This work was supported by the National Natural Science Foundation of the People's Republic of China, no. 18975001.

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